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On separably injective Banach spaces

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Abstract

We deal with two weak forms of injectivity which turn out to have a rich structure behind: separable injectivity and universal separable injectivity. We show several structural and stability properties of these classes of Banach spaces. We provide natural examples of (universally) separably injective spaces, including $\mathcal{L}_\infty$ ultraproducts built over countably incomplete ultrafilters, in spite of the fact that these ultraproducts are never injective. We obtain two fundamental characterizations of universally separably injective spaces. (a) A Banach space E is universally separably injective if and only if every separable subspace is contained in a copy of ℓ_∞ inside E. (b) A Banach space E is universally separably injective if and only if for every separable space S one has $\operatorname{Ext}(\ell_\infty/S, E) = 0$. Section 6 focuses on special properties of 1-separably injective spaces. Lindenstrauss proved in the middle sixties that, under CH, 1-separably injective spaces are 1-universally separably injective and left open the question in ZFC. We construct a consistent example of a Banach space of type $C(K)$ which is 1-separably injective but not universally 1-separably injective.

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1. Introduction

A Banach space E is said to be injective if for every Banach space X and every subspace Y of X, each operator $t: Y \to E$ admits an extension $T: X \to E$. The space E is said to be λ-injective if, moreover, T can be chosen so that $\|T\| \leq \lambda\|t\|$. The space ℓ_∞ is the best known example of 1-injective space. The two basic facts about injective spaces are that 1-injective spaces are isometric to $C(K)$-spaces with K extremely disconnected and that is not known if all injective spaces are isomorphic to 1-injective spaces.

In this paper we deal with two weak forms of injectivity which turn out to have a rich structure behind: separable injectivity and universal separable injectivity. A Banach space E is said to be separably injective if it satisfies the extension property in the definition of injective spaces under the restriction that X is separable; and it is said to be universally separably injective if it satisfies the extension property when Y is separable. Obviously, injective spaces are universally separably injective and these, in turn, are separably injective; the converse implications fail. We will mainly be concerned with nonseparable Banach spaces since c_0 is the only infinite dimensional separable separably injective Banach space, according to a deep result of Zippin [56]; we refer to [57] for what is perhaps the simplest proof.

The basic structural and stability properties of these classes are studied in Section 3: we will show that infinite-dimensional separably injective spaces are $\mathcal{L}_\infty$-spaces, contain c_0 and have Pełczyński's property (V). Universally separably injective spaces, moreover, are Grothendieck spaces, contain ℓ_∞ and enjoy a stronger form of Pełczyński's property (V) that we call Rosenthal's property (V). In Section 4 we provide natural examples of (universally) separably injective spaces, including the remarkable fact that ultraproducts built over countably incomplete ultrafilters are universally separably injective as long as they are $\mathcal{L}_\infty$-spaces, in spite of the fact that they are never injective. In Section 5 we obtain two fundamental characterizations of universally separably injective spaces: (a) a Banach space E is universally separably injective if and only if every separable subspace is contained in a copy of ℓ_∞ inside E. (b) A Banach space E is universally separably injective if and only if for every separable space S one has $\mathrm{Ext}(\ell_\infty/S, E) = 0$; i.e., E is complemented in any superspace Z such that $Z/E = \ell_\infty/S$. Characterization (a) allows to prove that universal separable injectivity is a 3-space property, which provides many new examples of spaces with this property. Characterization (b) leads to the result $\mathrm{Ext}(\ell_\infty/c_0, \ell_\infty/c_0) = 0$, which provides a new unexpected solution for equation $\mathrm{Ext}(X, X) = 0$. Section 6 focuses on special properties of 1-separably injective spaces. This is the point in which set theory axioms enter the game. Indeed, Lindenstrauss obtained in the middle sixties what can be understood as a proof that, under the continuum hypothesis, 1-separably injective spaces are 1-universally separably injective; he left open the question in ZFC. We construct a consistent example of a Banach space of type $C(K)$ which is 1-separably injective but not 1-universally separably injective. The final section contains several applications and open problems.

2. Background

Our notation is fairly standard, as in [44]. We will work with real scalars, although our results can be extended, *mutatis mutandis*, for complex scalars. Given a set Γ we denote by $\ell_\infty(\Gamma)$ the space of all bounded scalar functions on Γ, endowed with the sup norm and $c_0(\Gamma)$ is the closed subspace spanned by the characteristic functions of the singletons of Γ. A Banach space X is said to be a $\mathcal{L}_{\infty,\lambda}$-space if every finite dimensional subspace F of X is contained in another finite dimensional subspace of X whose Banach–Mazur distance to the corresponding ℓ_∞^n is at most λ. A space is said to be a $\mathcal{L}_\infty$-space if it is a $\mathcal{L}_{\infty,\lambda}$-space for some $\lambda \geq 1$; we will say that it is a $\mathcal{L}_{\infty,\lambda+}$-space when it is a $\mathcal{L}_{\infty,\lambda'}$-space for all $\lambda' > \lambda$. A Lindenstrauss space is one whose dual is isometric to $L_1(\mu)$ for some measure μ. Lindenstrauss spaces and $\mathcal{L}_{\infty,1+}$-spaces are identical classes.

We say that a Banach space is a $\mathcal{C}$-space if it is isometrically isomorphic to $C(K)$ for some compact space K. For a neat characterization of those real Banach algebras which are isometrically isomorphic to $C(K)$, see [1, Theorem 4.2.5].

Throughout the paper, ZFC denotes the usual setting of set theory with the Axiom of Choice, while CH denotes the continuum hypothesis.

2.1. *The push-out and pull-back constructions*

The push-out construction appears naturally when one considers a couple of operators defined on the same space, in particular in any extension problem. Let us explain why. Given operators $\alpha : Y \to A$ and $\beta : Y \to B$, the associated push-out diagram is

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & A \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\ B & \xrightarrow{\alpha'} & \mathrm{PO} \end{array} \tag{1}$$

Here, the push-out space $\mathrm{PO} = \mathrm{PO}(\alpha, \beta)$ is quotient of the direct sum $A \oplus_1 B$ (with the sum norm) by the closure of the subspace $\Delta = \{(\alpha y, -\beta y) : y \in Y\}$. The map α' is given by the inclusion of B into $A \oplus_1 B$ followed by the natural quotient map $A \oplus_1 B \to (A \oplus_1 B)/\overline{\Delta}$, so that $\alpha'(b) = (0, b) + \overline{\Delta}$ and, analogously, $\beta'(a) = (a, 0) + \overline{\Delta}$. The diagram (1) is commutative: $\beta'\alpha = \alpha'\beta$. Moreover, it is 'minimal' in the sense of having the following universal property: if $\beta'' : A \to C$ and $\alpha'' : B \to C$ are operators such that $\beta''\alpha = \alpha''\beta$, then there is a unique operator $\gamma : \mathrm{PO} \to C$ such that $\alpha'' = \gamma\alpha'$ and $\beta'' = \gamma\beta'$. Clearly, $\gamma((a, b) + \overline{\Delta}) = \beta''(a) + \alpha''(b)$ and one has $\|\gamma\| \leq \max\{\|\alpha''\|, \|\beta''\|\}$. The basic properties of push-out diagrams are gathered below; see [7, Lemma 2.1].

Lemma 2.1.

(a) $\max\{\|\alpha'\|, \|\beta'\|\} \leq 1$.
(b) *If α is an isomorphic embedding, then Δ is closed.*
(c) *If α is an isometric embedding and $\|\beta\| \leq 1$ then α' is an isometric embedding.*
(d) *If α is an isomorphic embedding then α' is an isomorphic embedding.*
(e) *If $\|\beta\| \leq 1$ and α is an isomorphism then α' is an isomorphism and $\|(\alpha')^{-1}\| \leq \max\{1, \|\alpha\|\}$.*

The pull-back construction is the dual of that of push-out in the sense of categories, that is, "reversing arrows". Indeed, let $\alpha : A \to Z$ and $\beta : B \to Z$ be operators acting between Banach spaces. The associated pull-back diagram is

$$\begin{array}{ccc} \mathrm{PB} & \xrightarrow{'\beta} & A \\ {\scriptstyle '\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ B & \xrightarrow{\beta} & Z \end{array} \tag{2}$$

Here, the pull-back space is $\mathrm{PB} = \mathrm{PB}(\alpha, \beta) = \{(a, b) \in A \oplus_\infty B : \alpha(a) = \beta(b)\}$. The arrows after primes are the restriction of the projections onto the corresponding factor. Needless to say (2) is minimally commutative in the sense that if the operators $''\beta : C \to A$ and $''\alpha : C \to B$ satisfy $\alpha \circ {''\beta} = \beta \circ {''\alpha}$, then there is a unique operator $\gamma : C \to \mathrm{PB}$ such that $''\alpha = {'\alpha}\gamma$ and $''\beta = {'\beta}\gamma$. Clearly, $\gamma(c) = (''\beta(c), ''\alpha(c))$ and $\|\gamma\| \le \max\{\|''\alpha\|, \|''\beta\|\}$. Quite clearly $'\alpha$ is onto if α is.

2.2. *Exact sequences*

A short exact sequence of Banach spaces is a diagram

$$0 \longrightarrow Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \longrightarrow 0 \tag{3}$$

where Y, X and Z are Banach spaces and the arrows are operators in such a way that the kernel of each arrow coincides with the image of the preceding one. By the open mapping theorem $\imath$ embeds Y as a closed subspace of X and Z is isomorphic to the quotient $X/\imath(Y)$. We say that $0 \to Y \to X_1 \to Z \to 0$ is equivalent to (3) if there exists an operator $t : X \to X_1$ making commutative the diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "t"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & X_1 \arrow[r] & Z \arrow[r] & 0. \end{tikzcd} \tag{4}$$

This is a true equivalence relation since such a t has to be an isomorphism. The sequence (3) is said to be trivial if it is equivalent to the direct sum sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. This happens if and only if it *splits*, that is, there is an operator $p : X \to Y$ such that $p\imath = \mathbf{1}_Y$ ($\imath(Y)$ is complemented in X); equivalently, there is an operator $s : Z \to X$ such that $\pi s = \mathbf{1}_Z$. For every pair of Banach spaces Z and Y, we denote by $\mathrm{Ext}(Z, Y)$ the space of all exact sequences $0 \to Y \to X \to Z \to 0$ modulo equivalence. We write $\mathrm{Ext}(Z, Y) = 0$ to indicate that every sequence of the form (3) is trivial. The reason for this notation is that $\mathrm{Ext}(Z, Y)$ has a natural linear structure [18,24] for which the (class of the) trivial extension is the zero element.

Suppose we are given an exact sequence (3) and an operator $t : Y \to B$. Consider the push-out of the couple $(\imath, t)$ and draw the corresponding arrows:

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "\imath"] \arrow[d, "t"'] & X \arrow[r, "\pi"] \arrow[d, "t'"] & Z \arrow[r] & 0 \\ & B \arrow[r, "\imath'"] & \mathrm{PO} & & \end{tikzcd}$$

By Lemma 2.1(a), $\imath'$ is an isomorphic embedding. Now, the operator $\pi : X \to Z$ and the zero operator $0 : B \to Z$ satisfy the identity $\pi\imath = 0t = 0$, and thus the universal property of the push-out gives a unique operator $\varpi : \mathrm{PO} \to Z$ – defined as $\varpi((x, b) + \Delta) = \pi(x)$ – making the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "\imath"] \arrow[d, "t"'] & X \arrow[r, "\pi"] \arrow[d, "t'"] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, "\imath'"] & \mathrm{PO} \arrow[r, "\varpi"] & Z \arrow[r] & 0 \end{tikzcd} \tag{5}$$

Elementary considerations show that the lower sequence in the preceding diagram is exact: check commutativity, and discard everything but the definition of PO. That sequence will we referred to as the push-out sequence. Actually, the universal property of the push-out makes this diagram unique, in the sense that for any other commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & {\scriptstyle t}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & X' & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

the lower row turns out to be equivalent to the push-out sequence in (5). For this reason we usually refer to a diagram like that as a push-out diagram. The universal property of the push-out diagram immediately yields the following.

Lemma 2.2. *With the above notations, the push-out sequence splits if and only if t extends to X, that is, there is an operator $T : X \to B$ such that $T\imath = t$.*

Proceeding dually one obtains the pull-back sequence. Consider again (3) and an operator $u : A \to Z$. Let us form the pull-back diagram of the couple (π, u) thus:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & & & {\scriptstyle 'u}\uparrow & & \uparrow{\scriptstyle u} & & \\ & & & & \mathrm{PB} & \xrightarrow{'\pi} & A & & \end{array}$$

Recalling that $'\pi$ is onto and taking $\jmath(y) = (0, \imath(y))$, it is easily seen that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \\ & & \| & & {\scriptstyle 'u}\uparrow & & \uparrow{\scriptstyle u} & & \\ 0 & \longrightarrow & Y & \xrightarrow{\jmath} & \mathrm{PB} & \xrightarrow{'\pi} & A & \longrightarrow & 0. \end{array} \tag{6}$$

The lower sequence is exact, and we shall referred to it as the pull-back sequence. The splitting criterion is now as follows.

Lemma 2.3. *With the above notations, the pull-back sequence splits if and only if u lifts to X, that is, there is an operator $U : A \to X$ such that $\pi U = u$.*

Given an exact sequence $0 \longrightarrow Y \xrightarrow{\imath} X \xrightarrow{\pi} Z \longrightarrow 0$ and another Banach space B, taking operators with values in B one gets the exact sequence

$$0 \longrightarrow \mathfrak{L}(Z, B) \xrightarrow{\pi^*} \mathfrak{L}(X, B) \xrightarrow{\imath^*} \mathfrak{L}(Y, B)$$

that can be continued to form a “long exact sequence”

$$0 \longrightarrow \mathfrak{L}(Z, B) \xrightarrow{\pi^*} \mathfrak{L}(X, B) \xrightarrow{\imath^*} \mathfrak{L}(Y, B) \xrightarrow{\beta} \mathrm{Ext}(Z, B) \longrightarrow \mathrm{Ext}(X, B)$$
$$\longrightarrow \mathrm{Ext}(Y, B).$$

Here we just indicate the action of β: it takes $t \in \mathfrak{L}(Y, B)$ into (the class in $\mathrm{Ext}(Z, B)$ of) the lower extension of the push-out diagram (5). A detailed description of homology sequences can be seen in [18].

3. Basic properties of (universally) separably injective spaces

Definition 3.1. A Banach space E is separably injective if for every separable Banach space X and each subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : X \to E$. If some extension T exists with $\|T\| \leq \lambda\|t\|$ we say that E is λ-separably injective.

It is easy to check that every separably injective space E is λ-separably injective for some λ since every sequence of norm-one operators $t_n : Y_n \to E$ induces a norm-one operator $t : \ell_1(Y_n) \to E$. Separable injective spaces can be characterized as follows.

Proposition 3.2. *For a Banach space E the following properties are equivalent.*

(a) *E is separably injective.*
(b) *Every operator from a subspace of ℓ_1 into E extends to ℓ_1.*
(c) *For every Banach space X and each subspace Y such that X/Y is separable, every operator $t : Y \to E$ extends to X.*
(d) *If X is a Banach space containing E and X/E is separable, then E is complemented in X.*
(e) *For every separable space S one has* $\mathrm{Ext}(S, E) = 0$.

Moreover,

(1) *The space E is λ-complemented in every Z such that Z/E is separable if and only if every operator $t : Y \to E$ admits an extension $T : X \to E$ with $\|T\| \leq \lambda\|t\|$, whenever X/Y is separable.*
(2) *If E is λ-separably injective, then for every operator $t : Y \to E$ there exists an extension $T : X \to E$ of T with $\|T\| \leq 3\lambda\|t\|$, whenever X/Y is separable.*

Proof. It is clear that (c) $\Rightarrow$ (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) $\Leftrightarrow$ (e). Moreover, (1) shows that (d) $\Rightarrow$ (c) and (2) shows that (a) $\Rightarrow$ (c). The remaining implication (b) $\Rightarrow$ (a) follows from the proof of (2) below.

For the sufficiency statement in (1) simply consider t as the identity on E. For the necessity statement, given an operator $t : Y \to E$ from the associated push-out diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "\imath"] \arrow[d, "t"'] & X \arrow[r, "\pi"] \arrow[d, "t'"] & X/Y \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E \arrow[r, "\imath'"] & \mathrm{PO} \arrow[r] & \mathrm{PO}/E \arrow[r] & 0. \end{tikzcd}$$

Since $\mathrm{PO}/E = X/Y$ is separable, there is a projection $p : \mathrm{PO} \to E$ with norm at most λ, and thus, recalling that $\|t'\| \leq 1$, the composition $pt' : X \to E$ yields an extension of t with norm at most λ.

The proof for (2) is a little more tricky. Let q be a surjective map from $\ell_1 \to X/Y$. The lifting property of ℓ_1 provides an operator $Q : \ell_1 \to X$. Consider thus the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker q \arrow[r, "\jmath"] \arrow[d, "\phi"'] & \ell_1 \arrow[r, "q"] \arrow[d, "Q"'] & X/Y \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & X \arrow[r] & X/Y \arrow[r] & 0 \end{tikzcd}$$

Let us construct the true push-out of the couple $(\phi, \jmath)$ and the corresponding complete diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker q & \xrightarrow{\jmath} & \ell_1 & \xrightarrow{q} & X/Y & \longrightarrow & 0 \\ & & \phi\downarrow & & \downarrow\phi' & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{\jmath'} & \mathrm{PO} & \longrightarrow & X/Y & \longrightarrow & 0. \end{array}$$

We can consider without loss of generality that $\|\phi\| = 1$. Let $S : \ell_1 \to E$ be an extension of $t\phi$ with $\|S\| \le \lambda\|t\phi\| \le \lambda\|t\|$. By the universal property of the push-out, there exists an operator $L : \mathrm{PO} \to E$ such that $L\phi' = S$ and $\|L\| \le \max\{\|t\|, \|S\|\} \le \lambda\|t\|$. Again by the universal property of the push-out, there is a diagram of equivalent exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\jmath'} & \mathrm{PO} & \longrightarrow & X/Y & \longrightarrow & 0 \\ & & \| & & \gamma\downarrow & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{p} & X/Y & \longrightarrow & 0, \end{array}$$

where the isomorphism γ is defined as $\gamma((y, u) + \Delta) = \jmath(y) + Q(u)$ is such that $\|\gamma\| \le \max\{\|\jmath\|, \|Q\|\} \le 1$. The desired extension of t to X is $T = L\gamma^{-1}$, where γ^{-1} comes defined by

$$\gamma^{-1}(x) = (x - s(px), s(px)) + \Delta,$$

where $s : X/Y \to \ell_1$ is a homogeneous bounded selection for q with $\|s\| \le 1$. One clearly has $\|\gamma^{-1}\| \le 3$, and therefore $\|T\| \le 3\lambda$. □

We are especially interested in the following subclass of separably injective spaces.

Definition 3.3. A Banach space E is said to be universally separably injective if for every Banach space X and each separable subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : X \to E$. If some extension T exists with $\|T\| \le \lambda\|t\|$ we say that E is universally λ-separably injective.

It is easy to check that a Banach space E is universally separably injective if and only if every E-valued operator with separable range extends to any superspace. It is also easy to show that every universally separably injective space is λ-universally separably injective for some λ.

Recall that a Banach space X has Pełczyński's property (V) if every operator defined on X is either weakly compact or it is an isomorphism on a subspace isomorphic to c_0. It is well-known that Lindenstrauss spaces (i.e., $\mathcal{L}_{\infty,1+}$-spaces) have this property [39]. Not all $\mathcal{L}_\infty$-spaces have Pełczyński's property (V): for example, the $\mathcal{L}_\infty$-spaces without copies of c_0 constructed by Bourgain and Delbaen [14]; or those that can be obtained from Bourgain–Pisier [15]; or the space Ω constructed in [21] as a twisted sum $0 \to C[0, 1] \to \Omega \to c_0 \to 0$ with strictly singular quotient map. Other examples appear in the more recent papers [6,32].

Definition 3.4. We will say that X has *Rosenthal's property* (V) if every operator defined on X is either weakly compact or it is an isomorphism on a subspace isomorphic to ℓ_∞.

Recall that a Banach space X is said to be a Grothendieck space if every operator from X to a separable Banach space (or to c_0) is weakly compact. Clearly, a Banach space with Pełczyński's property (V) is a Grothendieck space if and only if it has no complemented subspace isomorphic to c_0. It is well-known that ℓ_∞ is a Grothendieck space. One moreover has the following.

Proposition 3.5.

(a) *A separably injective space is of type $\mathcal{L}_\infty$, has Pełczyński's property (V) and, when it is infinite dimensional, contains copies of c_0.*
(b) *A universally separably injective space is a Grothendieck space of type $\mathcal{L}_\infty$, has Rosenthal's property (V) and, when it is infinite dimensional, contains ℓ_∞.*

Proof. (a) Let E be a λ-separably injective space. We want to see that if Y is a subspace of any Banach space X, every operator $t : Y \to E$ extends to an operator $T : X \to E^{**}$ with $\|T\| \leq \lambda\|t\|$. This implies that E^{**} is λ-injective, by an old result of Lindenstrauss [42, Theorem 2.1]. Since E^{**} is infinite-dimensional, it is then a $\mathcal{L}_{\infty,9\lambda^+}$ space and so is E. Let $t : Y \to E$ be an operator. Given a finite-dimensional subspace F of X, let $T_F : F \to E$ be any operator extending the restriction of t to $Y \cap F$. Let $\mathscr{F}$ be the set of finite-dimensional subspaces of X, ordered by inclusion, let $\mathscr{U}$ be any ultrafilter refining the Fréchet filter on $\mathscr{F}$, that is, containing every set of the form $\{G \in \mathscr{F} : F \subset G\}$ for fixed $F \in \mathscr{F}$. Then, define $T : X \to E^{**}$ taking

$$T(x) = \text{weak*-}\lim_{\mathscr{U}(F)} T_F(1_{F(x)}x).$$

It is easily seen that T is a linear extension of t, with $\|T\| \leq \lambda\|t\|$.

To show that E contains c_0 and has property (V), let $T : E \to X$ be a non-weakly compact operator (E being an infinite dimensional $\mathcal{L}_\infty$ space that cannot be reflexive). Choose a bounded sequence (x_n) in E such that (Tx_n) has no weakly convergent subsequences and let Y be the subspace spanned by (x_n) in E. As Y is separable we can regard it as a subspace of $C[0, 1]$. Let $J : C[0, 1] \to E$ be any operator extending the inclusion of Y into E. Since $TJ : C[0, 1] \to E$ is not weakly compact, TJ is an isomorphism on some subspace isomorphic to c_0; and the same occurs to T.

(b) If, in addition to that, E is universally separably injective we may take $T : E \to Z$ and $Y \subset E$ as before but this time we consider Y as a subspace of ℓ_∞. If $J : \ell_\infty \to E$ is any extension of the inclusion of Y into E, then $TJ : \ell_\infty \to Z$ is not weakly compact. Hence it is an isomorphism on some subspace isomorphic to ℓ_∞ and so is T. $\square$

Several modifications on the proof of Ostrovskii [48] yield the following.

Proposition 3.6. *A λ-separably injective space with $\lambda < 2$ is either finite-dimensional or has density character at least $\mathfrak{c}$.*

Proof. Let X be an infinite dimensional λ-separably injective space for $\lambda < 2$. We have shown that X contains c_0, and thus by a result of James [36] it contains, for each $\varepsilon > 0$, an $(1 + \varepsilon)$-isomorphic copy of c_0. With a standard renorming [50, Proposition 1] we may assume X contains c_0 isometrically and it is λ'-separably injective, still with $\lambda' < 2$. So, let $u : c_0 \to X$ be an isometric embedding and let $u_n = u(e_n)$, where (e_n) is the unit basis of c_0. Consider the push-out space

$$\begin{array}{ccc} c_0 & \xrightarrow{u} & X \\ {\scriptstyle\iota}\downarrow & & \downarrow \\ \ell_\infty & \longrightarrow & \mathrm{PO} \end{array}$$

in which ι is the natural inclusion map. In the rest of the proof we identify the elements of c_0, ℓ_∞ and X with their images in PO. For each element $f \in \ell_\infty$ with all coordinates ± 1, let

$P_f : X + [f] \to X$ be a projection with norm at most λ'. If $\operatorname{dens} X < \mathfrak{c}$, then, for each $\varepsilon > 0$, there exist $f \neq g$ such that $\|P_f(f) - P_g(g)\| < \varepsilon$. Pick n so that $f(n) = 1$ and $g(n) = -1$. One has $\|u_n - \frac{1}{2}f\|_{\mathrm{PO}} = \|u_n + \frac{1}{2}g\|_{\mathrm{PO}} = \frac{1}{2}$. This yields

$$\begin{aligned} 2 = 2\|u_n\| &< \|u_n + P_f(f/2) - P_g(g/2) + u_n\| + \varepsilon \\ &\leq \|P_f(u_n - (f/2))\| + \|P_g(u_n + (g/2))\| + \varepsilon \\ &\leq \|P_f\|\|u_n - f/2\| + \|P_g\|\|u_n + g/2\| + \varepsilon \\ &\leq \lambda' + \varepsilon < 2, \end{aligned}$$

a contradiction. □

Recall that a class of Banach spaces is said to have the 3-space property if whenever X/Y and Y belong to the class, then so X does. The monograph [24] contains a rather complete study of 3-space properties.

Proposition 3.7. (1) *The class of separably injective spaces has the* 3*-space property.*
(2) *The quotient of two separably injective spaces is separably injective.*
(3) *The class of universally separably injective spaces has the* 3*-space property.*
(4) *The quotient of a universally separably injective space by a separably injective space is universally separably injective.*

Proof. The simplest proof for the 3-space property (1) follows from characterization (2) in Proposition 3.2: let us consider an exact sequence $0 \longrightarrow F \longrightarrow E \xrightarrow{\pi} G \longrightarrow 0$ in which both F and G are separably injective. Let $\phi : K \to E$ be an operator from a subspace $\imath : K \to \ell_1$ of ℓ_1; then $\pi\phi$ can be extended to an operator $\Phi : \ell_1 \to G$, which can in turn be lifted to an operator $\Psi : \ell_1 \to E$. The difference $\phi - \Psi\imath$ takes values in F and can thus be extended to an operator $e : \ell_1 \to F$. The desired operator is $\Psi + e$. (A proof that properties having the form $\operatorname{Ext}(X, S) = 0$ are always 3-space properties of X can be found in [18].)

To prove (2) and (4) let us consider an exact sequence $0 \longrightarrow F \longrightarrow E \xrightarrow{\pi} G \longrightarrow 0$ in which F is separably injective and E is (universally) separably injective. Let $\phi : Y \to G$ be an operator from a separable space Y which is a subspace of a separable (arbitrary) space X. Consider the pull-back diagram

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] & E \arrow[r, "q"] & G \arrow[r] & 0 \\ 0 \arrow[r] & F \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r, "Q"] \arrow[u, "\Phi"'] & Y \arrow[r] \arrow[u, "\phi"'] & 0 \end{tikzcd}$$

Since F is separably injective, the lower exact sequence splits, so Q has a selection operator $s : Y \to \mathrm{PB}$. By the injectivity assumption about E, there exists an operator $T : X \to E$ agreeing with Qs on Y. Then $qT : X \to G$ is the desired extension of ϕ.

The proof for (3) has to wait until Theorem 5.2 when a suitable characterization of universally separably injective spaces will be presented. □

Several variations of these results can be seen in [26]. It is obvious that if $(E_i)_{i\in I}$ is a family of λ-separably injective Banach spaces, then $\ell_\infty(I, E_i)$ is λ-separably injective. The non-obvious fact that also $c_0(I, E_i)$ is separably injective can be considered as a vector valued version of Sobczyk's theorem. Proofs for this result have been obtained by Johnson–Oikhberg [38], Rosenthal [52], Cabello Sánchez [16] and Castillo–Moreno [25], each with its own estimate for the constant. These are $2\lambda^2$ (implicitly), $\lambda(1+\lambda)^+$, $(3\lambda^2)^+$ and $6(1+\lambda)$, respectively.

Remark 3.8. Let $0 \to F \to E \to G \to 0$ be a short exact sequence of Banach spaces. We know from Proposition 3.7 that E is separably injective if the other two relevant spaces are; and the same happens with G. What about F? Bourgain [12] constructed an uncomplemented copy of ℓ_1 in ℓ_1, which yields an exact sequence $0 \to \ell_1 \to \ell_1 \to B \to 0$ that does not split. By Lindenstrauss' lifting B is not a $\mathcal{L}_1$ space. Its dual sequence $0 \to B^* \to \ell_\infty \to \ell_\infty \to 0$ shows that the kernel of a quotient mapping between two injective spaces may fail to be even a $\mathcal{L}_\infty$-space.

4. Examples

All injective spaces are universally separably injective. Sobczyk theorem states that c_0 is 2-separably injective in its natural supremum norm and so is and $c_0(\Gamma)$ for any index set Γ (see [22] for a rather detailed survey). They are not universally separably injective since they do not contain ℓ_∞.

4.1. Twisted sums

The 3-space property yields that twisted sums of separably injective are also separably injective. In particular:

- Twisted sums of c_0 and $c_0(\Gamma)$. This includes the Johnson–Lindenstrauss spaces $JL_\infty(\mathscr{M})$ [37] obtained by taking the closure of the linear span in ℓ_∞ of c_0 and the characteristic functions of the sets of an uncountable almost disjoint family $\mathscr{M}$ of subsets of $\mathbb{N}$. Marciszewski and Pol answer in [47] a question of Koszmider [40, Question 5] showing that there exist $2^{\mathfrak{c}}$ almost disjoint families $\mathscr{M}$ generating mutually non-isomorphic Johnson–Lindenstrauss spaces, all of them $\mathcal{C}$-spaces.
- Twisted sums of two nonseparable $c_0(\Gamma)$ spaces. This includes variations of the previous construction using the Sierpinski–Tarski [41] generalization of the construction of almost-disjoint families; the Ciesielski–Pol space (see [29]); the WCG nontrivial twisted sums of $c_0(\Gamma)$ obtained independently by Argyros, Castillo, Granero, Jimenez and Moreno [5] and by Marciszewski [46].
- Twisted sums of c_0 and ℓ_∞, as those constructed in [20].
- Benyamini constructed in [11] an $\mathcal{M}$-space which is not complemented in any $\mathcal{C}$-space. This space turns out to be a twisted sum of c_0 and $c_0(\ell_\infty/c_0)$ (see [24, p. 104] or [26, Prop.3.3]) and so it is separably injective by Proposition 3.7(a).

It is not hard to prove that none of the preceding examples can be universally separably injective.

4.2. The space $\ell_\infty^c(\Gamma)$

A typical 1-universally separably injective space is the space $\ell_\infty^c(\Gamma)$ of countably supported bounded functions $f : \Gamma \to \mathbb{R}$, where Γ is an uncountable set. One has the following.

Proposition 4.1. *The space $\ell_\infty^c(\Gamma)$ is:* (a) *universally* 1*-separably injective;* (b) *not isometric to a $\mathcal{C}$-space;* (c) *isomorphic to a $\mathcal{C}$-space;* (d) *not injective.*

Proof. (a) Every separable subspace of $\ell_\infty^c(\Gamma)$ is contained in an isometric copy of ℓ_∞. (b) The unit ball of every $C(K)$ has extreme points and quite clearly, the ball of $\ell_\infty^c(\Gamma)$ has no extreme point. (c) Consider the unitization of $\ell_\infty^c(\Gamma)$ inside $\ell_\infty(\Gamma)$, that is,

$$\ell_\infty^c(\Gamma)_+ = \{f \in \ell_\infty(\Gamma) : f = \lambda 1_\Gamma + g : \lambda \in \mathbb{R}, g \in \ell_\infty^c(\Gamma)\}.$$

It is clear that $\ell_\infty^c(\Gamma)$ is 2-isomorphic to $\ell_\infty^c(\Gamma)_+$. As every unital subalgebra of $\ell_\infty(\Gamma)$ the latter is isometrically isomorphic to the algebra of all continuous real-valued functions on certain compact space K. (d) $\ell_\infty^c(\Gamma)$ contains $c_0(\Gamma)$. But Rosenthal proved in [51] that an injective space containing $c_0(I)$ must also contain $\ell_\infty(I)$. Thus, if $\ell_\infty^c(\Gamma)$ were injective, it should contain a subspace isomorphic to $\ell_\infty(\Gamma)$. This is obviously impossible if the cardinality of Γ satisfies $|\Gamma|^{\aleph_0} < 2^{|\Gamma|}$ by comparison of their density characters. But indeed the result holds even if $|\Gamma|$ does not satisfy that inequality. To see this, let us show that, given any uncountable Γ, there is no injective operator $T : \ell_\infty(\aleph_1) \longrightarrow \ell_\infty^c(\Gamma)$. With a slight abuse of notation, $\ell_\infty(\aleph_1)^*$ is the space of all bounded finitely additive measures on the family $\wp(\aleph_1)$ of all subsets of $\aleph_1$. Consider the measures $\nu_\gamma = T^*(\delta_\gamma)$, $\gamma \in \Gamma$, for an operator T as above. They have the property that for every $A \subset \aleph_1$, $\{\gamma \in \Gamma : \nu_\gamma(A) \neq 0\}$ is countable. Let $f : \aleph_1 \longrightarrow 2^{\mathbb{N}}$ be a one-to-one map. The set Δ of those $\gamma \in \Gamma$ for which there is a clopen $U \subset 2^{\mathbb{N}}$ with $\nu_\gamma(f^{-1}(U)) \neq 0$ is countable, by the above property and the fact that there are only countably many clopens. For all $\gamma \notin \Delta$ we have that $\nu_\gamma\{\alpha\} = 0$ for all $\alpha \in \aleph_1$. On the other hand, since Δ is countable, $\nu_\gamma\{\alpha\} = 0$ for all $\gamma \in \Delta$ and all but countably many α's in $\aleph_1$. In particular, there exist $\alpha \neq \beta, \alpha, \beta \in \aleph_1$ such that $\nu_\gamma\{\alpha\} = \nu_\gamma\{\beta\} = 0$ for all $\gamma \in \Gamma$. This means that $\delta_\gamma(T1_\alpha) = \delta_\gamma(T1_\beta)$ for all γ, that implies that $T1_\alpha = T1_\beta$ which contradicts that T is one-to-one. □

The space $\ell_\infty^c(\Gamma)$ shows that, contrarily to what happens in the injective case, universally 1-separably injective spaces need not to be isometric to any $\mathcal{C}$-space. Actually there exist universally 1-separably injective spaces which are not even complemented in any $\mathcal{C}$-space; see [7] for an account.

4.3. The space ℓ_∞/c_0

Since ℓ_∞ is injective and c_0 is separably injective, it follows from Proposition 3.7 that ℓ_∞/c_0 is universally separably injective, although the constant thus obtained is not optimal. It will follow from Proposition 4.6(a) that ℓ_∞/c_0 is 1-universally separably injective. Moreover, it can also be proved that every separable subspace of ℓ_∞/c_0 is contained in a subalgebra of ℓ_∞/c_0 isometrically isomorphic to ℓ_∞.

It is well-known that ℓ_∞/c_0 is not injective. The simplest proof appears in Rosenthal [51]: the space ℓ_∞/c_0 contains $c_0(\mathfrak{c})$ while it cannot contain $\ell_\infty(\mathfrak{c})$. This proof is quite rough in a sense: it says that ℓ_∞/c_0 is uncomplemented in its bidual, a huge superspace. Denoting $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, Amir had shown in [2] that $C(\mathbb{N}^*)$ is not complemented in $\ell_\infty(2^{\mathfrak{c}})$, which provides another proof that ℓ_∞/c_0 is not injective. It can be even shown [27] that $C(\mathbb{N}^*)$ contains an uncomplemented copy Y of itself. Proposition 5.6 yields that the corresponding quotient $C(\mathbb{N}^*)/Y$ cannot be isomorphic to a quotient of ℓ_∞ by a separable subspace.

4.4. Other $C(K)$-spaces

Recall that a compact Hausdorff space K is said to be an F-space if disjoint open F_σ sets (equivalently, cozeros) have disjoint closures. Equivalently, if any continuous function $f : K \to \mathbb{R}$ can be decomposed as $f = u|f|$ for some continuous function $u : K \to \mathbb{R}$. It is an immediate consequence of Tietze's extension theorem that a closed subset of an F-space is an F-space. In particular, $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ is an F-space. One has the following.

Proposition 4.2. *A compact space K is an F-space if and only if the Banach space $C(K)$ is* 1-*separably injective.*

The sufficiency is a particular case of the implication (ii) $\Rightarrow$ (iv) in [43]; while the necessity follows from (iv) $\Rightarrow$ (i) in the same paper. See also [9] for several generalized forms of this result. Simple examples show that K may fail to be an F-space if the space $C(K)$ is merely isomorphic even to a 1-injective space. This can be deduced from [54, Theorem 2.6], or else from [53]: identify two points $u, v \in \mathbb{N}^*$ that we may consider as two free ultrafilters $\mathscr{U}$ and $\mathscr{V}$ on $\mathbb{N}$ and let us call $\beta(u, v)$ the corresponding quotient space of $\beta\mathbb{N}$. The space $C(\beta(u, v)) = \{f \in C(\beta\mathbb{N}) : f(u) = f(v)\}$ is a closed hyperplane of $C(\beta\mathbb{N})$ and thus it is 2-isomorphic to ℓ_∞. However, $\beta(u, v)$ is not an F-space: pick $U \in \mathscr{U} \setminus \mathscr{V}$, so that $V = \mathbb{N} \setminus U$ belongs to $\mathscr{V}$. Set the function $f : \mathbb{N} \to \mathbb{R}$ given by

$$f(n) = \frac{1_U(n) - 1_V(n)}{n}$$

and extend it to a continuous function on $\beta\mathbb{N}$ denoted again f. As $f(u) = f(v) = 0$ we have $f \in C(\beta(u, v))$. However there is no factorization $f = g|f|$ with $g \in C(\beta(u, v))$ since in this case we would have $g(u) = 1$ and $g(v) = -1$.

Given a compact space K, we write K' for its derived set, that is, the set of its accumulation points. This process can be iterated to define $K^{(n+1)}$ as $(K^{(n)})'$. We say that K has height n if $K^{(n)} = \varnothing$. We say that K has finite height if it has height n for some $n \in \mathbb{N}$.

Proposition 4.3. *If K is a compact space of height n, then $C(K)$ is $(2n-1)$-separably injective. Consequently, if K is a compact space of finite height then $C(K)$ is separably injective although it is not universally separably injective unless K is finite.*

Proof. Let $Y \subset X$ with X separable and let $t : Y \to C(K)$ be a norm one operator. The range of t is separable and every separable subspace of a $C(K)$ is contained in an isometric copy of $C(L)$, where L is the quotient of K after identifying k and k' when $y(k) = y(k')$ for all $y \in Y$. This L is metrizable because Y is separable. Moreover, if K has height n, then L has height at most n and so it is homeomorphic to $[0, \omega^r \cdot k]$ with $r < n, k < \omega$. Since $C[0, \omega^r \cdot k]$ is $(2r + 1)$-separably injective [10], our operator can be extended to an operator $T : X \to C(K)$ with norm

$$\|T\| \le (2r + 1)\|t\| \le (2n - 1)\|t\|,$$

concluding the proof. $\square$

Using arguments from Amir [2], Baker [10] showed that $(2n - 1)$ is the best constant for separable injectivity when K is a metrizable compact of finite height n. There are some difficulties in generalizing those arguments for nonmetrizable compact spaces, so we do not know if it could exist a nonmetrizable compact space K of height n such that $C(K)$ is λ-separably injective for some $\lambda < 2n - 1$.

Proposition 4.4. *The space of all bounded Borel (respectively, Lebesgue) measurable functions on the line is* 1-*separably injective in the sup norm.*

Proof. The given spaces are (unital) closed subalgebras of $\ell_\infty(\mathbb{R})$. Thus they can be represented as $C(K)$ spaces. On the other hand, each measurable function can be decomposed as $f = u|f|$, with u (and $|f|$, of course) measurable. This clearly implies that the corresponding compacta are F-spaces. $\square$

Argyros proved in [4] that none of the spaces in the above example is injective. This is very simple in the Borel case: the characteristic functions of the singletons generate a copy of $c_0(\mathbb{R})$ in the space of bounded Borel functions. The density character of the latter space is the continuum, as there are $\mathfrak{c}$ Borel subsets. Therefore it cannot contain a copy of $\ell_\infty(\mathbb{R})$, whose density character is $2^{\mathfrak{c}}$.

4.5. M-ideals

A closed subspace $J \subset X$ is called an M-ideal [33, Definition 1.1] if its annihilator $J^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0 \text{ for all } x \in J\}$ is an L-summand in X^*. This just means that there is a linear projection P on X^* whose range is $J^\perp$ and such that $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. The easier examples of M-ideals are just ideals in $C(K)$-spaces. In particular, if H is a closed subset of the compact space K and $L = K \setminus H$ then $C_0(L)$ is an M-ideal in $C(K)$, which is straightforward from the Riesz representation of $C(K)^*$. A remarkable generalization of the Borsuk–Dugundji theorem for M-ideals was provided by Ando [3] and, independently, Choi and Effros [28]. In order to state it let us recall that a Banach space Z has the λ-approximation property (λ-AP) if, for every $\varepsilon > 0$ and every compact subset K of Z, there exists a finite rank operator T on Z, with $\|T\| \le \lambda$, such that $\|Tz - z\| < \varepsilon$, for every $z \in K$. We say that Z has the bounded approximation property (BAP) if it has the λ-AP, for some λ. We refer the reader to [23] for background and basic information about approximation properties.

Theorem 4.5 (*[33], Theorem 2.1*)**.** *Let J be an M-ideal in the Banach space E and $\pi : E \to E/J$ the natural quotient map. Let Y be a separable Banach space and $t : Y \to E/J$ be an operator. Assume further that one of the following conditions is satisfied.*

(1) *Y has the λ-AP.*
(2) *J is a Lindenstrauss space.*

Then t can be lifted to E, that is, there is an operator $T : Y \to E$ such that $\pi T = t$. Moreover one can get $\|T\| \le \lambda\|t\|$ under the assumption (1) *and $\|T\| = \|t\|$ under* (2).

One has the following.

Proposition 4.6. *Let J be an M-ideal in a Banach space E.*

(a) *If E is λ-(universally) separably injective, then E/J is λ^2-(universally) separably injective.*
(b) *If E is λ-separably injective, then J is $2\lambda^2$-separably injective.*

When J is a Lindenstrauss space (which is always the case if E is), then the exponent 2 *can be eliminated. In particular, if H is a closed subset of the compact space K and $L = K \setminus H$ one has the following.*

(c) *If $C(K)$ is λ-(universally) separably injective, then so is $C(H)$.*
(d) *If $C(K)$ is λ-separably injective, then $C_0(L)$ is 2λ-separably injective.*

Proof. (a) By (the proof of) Proposition 3.5, E^{**} is λ-injective and so it has the λ-AP. Since $E^{**} = J^{**} \oplus_\infty (E/J)^{**}$ we see that also J^{**} and $(E/J)^{**}$ have the λ-AP. Hence both J and (E/J) have the λ-AP. Let Y be a separable subspace of X and $t : Y \longrightarrow E/J$ an operator. Let S be a separable subspace of E/J containing the image of t. By [23, Theorem 9.7] we may assume S has the λ-AP. Let $s : S \longrightarrow E$ be the lifting provided by Theorem 4.5, so that $\|s\| \le \lambda$. Now, if $T : X \longrightarrow E$ is an extension of st, then $\pi T : X \longrightarrow E/J$ is an extension of t, and this can be achieved with $\|\pi T\| = \|T\| \le \lambda^2\|t\|$.

(d) – and (b) –. Let us remark that if S is a subspace of $C(K)$ containing $C_0(L)$ and $S/C_0(L)$ is separable, then there is a projection $p : S \longrightarrow C_0(L)$ of norm at most 2. Indeed, $S/C_0(L)$ is a separable subspace of $C(K_1)$ and there is a lifting $s : S/C_0(L) \longrightarrow C(K)$, with $\|s\| = 1$, and $p = id_S - sr$ is the required projection. Now, let $t : Y \longrightarrow C_0(L)$ be an operator, where Y is a subspace of a separable Banach space X. Considering t as taking values in $C(K)$, there is an extension $T : X \longrightarrow C(K)$ with $\|T\| \le \lambda\|t\|$. Let S denote the least closed subspace of $C(K)$ containing the range of T and $C_0(L)$ and $p : S \longrightarrow C_0(L)$ a projection with $\|p\| \le 2$. The composition $pT : X \longrightarrow C_0(L)$ is an extension of t and clearly, $\|pT\| \le 2\lambda\|t\|$. □

4.6. *Ultraproducts of type $\mathcal{L}_\infty$*

Let us briefly recall the definition and some basic properties of ultraproducts of Banach spaces. For a detailed study of this construction at the elementary level needed here we refer the reader to Heinrich's survey paper [34] or Sims' notes [55]. Let I be a set, $\mathscr{U}$ be an ultrafilter on I, and $(X_i)_{i\in I}$ a family of Banach spaces. Then $\ell_\infty(I, X_i)$, endowed with the supremum norm, is a Banach space, and $c_0^{\mathscr{U}}(X_i) = \{(x_i) \in \ell_\infty(I, X_i) : \lim_{\mathscr{U}(i)} \|x_i\| = 0\}$ is a closed subspace of $\ell_\infty(I, X_i)$. The ultraproduct of the family $(X_i)_{i\in I}$ following $\mathscr{U}$ is defined as the quotient

$$[X_i]_{\mathscr{U}} = \ell_\infty(X_i)/c_0^{\mathscr{U}}(X_i).$$

We denote by $[(x_i)]$ the element of $[X_i]_{\mathscr{U}}$ which has the family (x_i) as a representative. It is not difficult to show that $\|[(x_i)]\| = \lim_{\mathscr{U}(i)} \|x_i\|$. In the case $X_i = X$ for all i, we denote the ultraproduct by $X_{\mathscr{U}}$, and call it the ultrapower of X following $\mathscr{U}$. If $T_i : X_i \to Y_i$ is a uniformly bounded family of operators, the ultraproduct operator $[T_i]_{\mathscr{U}} : [X_i]_{\mathscr{U}} \to [Y_i]_{\mathscr{U}}$ is given by $[T_i]_{\mathscr{U}}[(x_i)] = [T_i(x_i)]$. Quite clearly, $\|[T_i]_{\mathscr{U}}\| = \lim_{\mathscr{U}(i)} \|T_i\|$.

Definition 4.7. An ultrafilter $\mathscr{U}$ on a set I is countably incomplete if there is a sequence (I_n) of subsets of I such that $I_n \in \mathscr{U}$ for all n, and $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

It is obvious that any countably incomplete ultrafilter is non-principal and also that every non-principal (or free) ultrafilter on $\mathbb{N}$ is countably incomplete. Assuming all free ultrafilters countably incomplete is consistent with ZFC, since the cardinal of a set supporting a free countably complete ultrafilter should be measurable, hence strongly inaccessible. From now on we will assume that all ultrafilters we use are countably incomplete.

The class of $\mathcal{L}_{\infty,\lambda+}$ spaces is stable under ultraproducts [13, Proposition 1.22]. In the opposite direction, a Banach space is a $\mathcal{L}_{\infty,\lambda+}$ space if and only if some (or every) ultrapower is. In particular, a Banach space is a Lindenstrauss space if and only if so are its ultrapowers. It is possible however to produce Lindenstrauss spaces as ultraproducts of families containing no $\mathcal{L}_\infty$-space. Indeed, if $p(i) \to \infty$ along $\mathscr{U}$, then the ultraproduct $[L_{p(i)}]_{\mathscr{U}}$ is a Lindenstrauss space (and, in fact, an abstract $\mathcal{M}$-space; see [17, Lemma 3.2]). The following result about the structure of separable subspaces of ultraproducts of type $\mathcal{L}_\infty$ will be fundamental for us.

Lemma 4.8. *Suppose $[X_i]_{\mathscr{U}}$ is a $\mathcal{L}_{\infty,\lambda+}$-space. Then each separable subspace of $[X_i]_{\mathscr{U}}$ is contained in a subspace of the form $[F_i]_{\mathscr{U}}$, where $F_i \subset X_i$ is finite dimensional and $\lim_{\mathscr{U}(i)} d(F_i, \ell_\infty^{k(i)}) \le \lambda$, with $k(i) = \dim F_i$.*

Proof. Let us assume S is an infinite-dimensional separable subspace of $[X_i]_{\mathscr{U}}$. Let (s^n) be a linearly independent sequence spanning a dense subspace in S and, for each n, let (s_i^n) be a fixed representative of s^n in $\ell_\infty(X_i)$. Let $S^n = \operatorname{span}\{s^1, \dots, s^n\}$. Since $[X_i]_{\mathscr{U}}$ is a $\mathcal{L}_{\infty,\lambda+}$-space there is, for each n, a finite dimensional $F^n \subset [X_i]_{\mathscr{U}}$ containing S^n with $d(F^n, \ell_\infty^{\dim F^n}) \le \lambda + 1/n$. For fixed n, let (f^m) be a basis for F^n containing $s^1, \dots, s^n$. Choose representatives (f_i^m) such that $f_i^m = s_i^\ell$ if $f^m = s^\ell$. Moreover, let F_i^n be the subspace of X_i spanned by f_i^m for $1 \le m \le \dim F^n$.

Let (I_n) be a decreasing sequence of subsets $I_n \in \mathscr{U}$ such that $\bigcap_{n=1}^\infty I_n = \varnothing$. For each integer n put

$$J_n' = \{i \in I : d(F_i^n, \ell_\infty^{\dim F^n}) \le \lambda + 2/n\} \cap I_n$$

and $J_m = \bigcap_{n \le m} J_n'$. All these sets are in $\mathscr{U}$ and $\bigcap_m J_m = \varnothing$. We define a function $k : I \to \mathbb{N}$ as

$$k(i) = \sup\{n : i \in J_n\}.$$

For each $i \in I$, take $F_i = F_i^{k(i)}$. This is a finite-dimensional subspace of X_i whose Banach–Mazur distance to the corresponding ℓ_∞^k is at most $\lambda + 2/k(i)$. It is clear that $[F_i]_{\mathscr{U}}$ contains S and also that $k(i) \to \infty$ along $\mathscr{U}$, which completes the proof. $\square$

Lemma 4.9. *For every function $k : I \to \mathbb{N}$, the space $[\ell_\infty^{k(i)}]_{\mathscr{U}}$ is universally 1-separably injective.*

Proof. Let Γ be the disjoint union of the sets $\{1, 2, \dots, k(i)\}$ viewed as a discrete set. Now observe that $c_0^{\mathscr{U}}(\ell_\infty^{k(i)})$ is an ideal in $\ell_\infty(\ell_\infty^{k(i)}) = \ell_\infty(\Gamma) = C(\beta\Gamma)$ and apply Proposition 4.6(a). $\square$

Theorem 4.10. *Let $(X_i)_{i\in I}$ be a family of Banach spaces such that $[X_i]_{\mathscr{U}}$ is a $\mathcal{L}_{\infty,\lambda^+}$-space. Then $[X_i]_{\mathscr{U}}$ is universally λ-separably injective.*

Proof. It is clear that a Banach space is universally λ-separably injective if and only if every separable subspace is contained in some larger universally λ-separably injective subspace. Combine now the last two lemmata. $\square$

Corollary 4.11. *Let $(X_i)_{i\in I}$ be a family of Banach spaces. If $[X_i]_{\mathscr{U}}$ is a Lindenstrauss space, then it is universally 1-separably injective.*

5. Two characterizations of universally separably injective spaces

In Proposition 3.5 (b) it was proved that universally separably injective spaces contain ℓ_∞. Much more is indeed true.

Definition 5.1. We say that a Banach space X is ℓ_∞-*upper-saturated* if every separable subspace of X is contained in some (isomorphic) copy of ℓ_∞ inside X.

It is easy to see that if X is ℓ_∞-upper-saturated, then there is $\lambda \ge 1$ so that every separable subspace of X is contained in a λ-isomorphic copy of ℓ_∞ inside X: if X_n is a separable subspace of X that can only be embedded into n-isomorphic copies of ℓ_∞ then $\bigcup X_n$ spans a separable subspace of X that cannot be embedded into any copy of ℓ_∞.

Theorem 5.2. *An infinite-dimensional Banach space is universally separably injective if and only if it is ℓ_∞-upper-saturated.*

Proof. The sufficiency is a consequence of the injectivity of ℓ_∞. In order to show the necessity, let Y be a separable subspace of a universally separably injective space X. We consider a subspace Y_0 of ℓ_∞ isomorphic to Y and an isomorphism $t : Y_0 \to Y$. We can find projections p on X and q on ℓ_∞ such that $Y \subset \ker p$, $Y_0 \subset \ker q$, and both p and q have range isomorphic to ℓ_∞. Indeed, let $\pi : X \to X/Y$ be the quotient map. Since X contains ℓ_∞ and Y is separable, π is not weakly compact so, by Proposition 3.5(b), there exists a subspace M of X isomorphic to ℓ_∞ where π is an isomorphism. Now $X/Y = \pi(M) \oplus N$, with N a closed subspace. Hence $X = M \oplus \pi^{-1}(N)$, and it is enough to take p as the projection with range M and kernel $\pi^{-1}(N)$. Since $\ker p$ and $\ker q$ are universally separably injective spaces, we can take operators $u : X \to \ker q$ and $v : \ell_\infty \to \ker p$ such that $v = t$ on Y_0 and $u = t^{-1}$ on Y.

Let $w : \ell_\infty \to \operatorname{ran} p$ be an operator satisfying $\|w(x)\| \geq \|x\|$ for all $x \in \ell_\infty$. We will show that the operator

$$T = v + w(\mathbf{1}_{\ell_\infty} - uv) : \ell_\infty \longrightarrow X$$

is an isomorphism (into). This suffices to end the proof since $\operatorname{ran} T$ is isomorphic to ℓ_∞ and both T and v agree with t on Y_0, so $Y \subset \operatorname{ran} T \subset X$. Since $\operatorname{ran} v \subset \ker p$ and $\operatorname{ran} w \subset \operatorname{ran} p$, there exists $C > 0$ such that for all $x \in \ell_\infty$ one has

$$\|Tx\| \geq C \max\{\|v(x)\|, \|w(\mathbf{1}_{\ell_\infty} - uv)x\|\}.$$

Now, if $\|vx\| < (2\|u\|)^{-1}\|x\|$, then $\|uvx\| < \frac{1}{2}\|x\|$; hence

$$\|w(\mathbf{1}_{\ell_\infty} - uv)x\| \geq \|(\mathbf{1}_{\ell_\infty} - uv)x\| > \frac{1}{2}\|x\|.$$

Thus $\|Tx\| \geq C(2\|u\|)^{-1}\|x\|$ for every $x \in X$. $\square$

We can now complete the proof of Proposition 3.7(3) and show that the class of universally separably injective spaces has the 3-space property.

Proposition 5.3. *The class of universally separably injective spaces has the* 3*-space property.*

Proof. By Theorem 5.2 one has to show that being ℓ_∞-upper-saturated is a 3-space property. Let $0 \longrightarrow Y \longrightarrow X \xrightarrow{q} Z \longrightarrow 0$ be an exact sequence in which both Y, Z are ℓ_∞-upper-saturated, and let S be a separable subspace of X. It is not hard to find separable subspaces S_0, S_1 of X such that $S \subset S_1$ and $S_1/S_0 = [q(S)]$. Let Y_∞ be a copy of ℓ_∞ inside Y containing S_0. By the injectivity of ℓ_∞, S is contained in the subspace $Y_\infty \oplus [q(S)]$ of X. And since there exists a copy Z_∞ of ℓ_∞ containing $[q(S)]$, S is therefore contained in the subspace $Y_\infty \oplus Z_\infty$ of X, which is isomorphic to ℓ_∞. $\square$

A homological characterization of universally separably injective spaces is also possible. We need first to show the following.

Proposition 5.4. *If U is a universally separably injective space then* $\operatorname{Ext}(\ell_\infty, U) = 0$.

Proof. Partington's distortion theorem [49, Theorem 3] asserts that a Banach space containing ℓ_∞ contains an almost isometric copy of ℓ_∞ (see also Dowling [31]). This last copy will therefore be, say, 2-complemented. Let Γ denote the set of all the 2-isomorphic copies of ℓ_∞ inside ℓ_∞. For each $E \in \Gamma$ let $\iota_E : E \to \ell_\infty$ be the canonical embedding, p_E a projection onto E of norm at most 2 and $u_E : E \to \ell_\infty$ an isomorphism, with $\|u_E\|\|u_E^{-1}\| \leq 2$. Assume that a nontrivial exact sequence

$$0 \longrightarrow U \longrightarrow X \longrightarrow \ell_\infty \longrightarrow 0$$

exists. We consider, for each $E \in \Gamma$, a copy of the preceding sequence, and form the product of all these copies $0 \longrightarrow \ell_\infty(\Gamma, U) \longrightarrow \ell_\infty(\Gamma, X) \longrightarrow \ell_\infty(\Gamma, \ell_\infty) \longrightarrow 0$. Let us consider the embedding $J : \ell_\infty \to \ell_\infty(\Gamma, \ell_\infty)$ defined as $J(x)(E) = u_E p_E(x)$ and then form the pull-back sequence

$$\begin{tikzcd}
0 \arrow[r] & \ell_\infty(\Gamma, U) \arrow[r] & \ell_\infty(\Gamma, X) \arrow[r] & \ell_\infty(\Gamma, \ell_\infty) \arrow[r] & 0 \\
0 \arrow[r] & \ell_\infty(\Gamma, U) \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r, "q"] \arrow[u] & \ell_\infty \arrow[r] \arrow[u, "J"'] & 0.
\end{tikzcd}$$

Let us show that q cannot be an isomorphism on a copy of ℓ_∞. Otherwise, it would be an isomorphism on some $E \in \Gamma$ and thus the new pull-back sequence

$$\begin{tikzcd}
0 \arrow[r] & \ell_\infty(\Gamma, U) \arrow[r] & \mathrm{PB} \arrow[r, "q"] & \ell_\infty \arrow[r] & 0 \\
0 \arrow[r] & \ell_\infty(\Gamma, U) \arrow[r] \arrow[u, equal] & \mathrm{PB}_E \arrow[r] \arrow[u] & E \arrow[r] \arrow[u, "\iota_E"'] & 0
\end{tikzcd}$$

would split. And therefore the same would be true making push-out with the canonical projection $\pi_E : \ell_\infty(\Gamma, U) \to U$ onto the E-th copy of U:

$$\begin{tikzcd}
0 \arrow[r] & \ell_\infty(\Gamma, U) \arrow[r] \arrow[d, "\pi_E"'] & \mathrm{PB}_E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \arrow[r] & \mathrm{PO}_E \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

But it is not hard to see that new pull-back with u_E^{-1}

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] & \mathrm{PO}_E \arrow[r] & E \arrow[r] & 0 \\
0 \arrow[r] & U \arrow[r] \arrow[u, equal] & X \arrow[r] \arrow[u] & \ell_\infty \arrow[r] \arrow[u, "u_E^{-1}"'] & 0
\end{tikzcd}$$

produces exactly the starting sequence which, by assumption, was nontrivial.

However, the space PB should be universally separably injective by Proposition 3.7(3); hence it must have Rosenthal's property (V), by Proposition 3.5(b), and therefore q should be an isomorphism on some copy of ℓ_∞. This contradiction shows that the starting nontrivial sequence cannot exist. □

We are thus ready to establish our second characterization of universal separable injectivity. Please note that every separable Banach space embeds into ℓ_∞ in exactly one way up to automorphisms of ℓ_∞, by the Lindenstrauss–Rosenthal theorem [44, Theorem 2.f.12].

Theorem 5.5. *A Banach space U is universally separably injective if and only if for every separable Banach space S one has* $\mathrm{Ext}(\ell_\infty/S, U) = 0$.

Proof. Let S be a separable subspace of ℓ_∞ and let U be universally separably injective. The homology sequence obtained by applying $\mathfrak{L}(-, U)$ to the sequence $0 \to S \to \ell_\infty \to \ell_\infty/S \to 0$ yields an exact sequence

$$\cdots \longrightarrow \mathfrak{L}(\ell_\infty, U) \longrightarrow \mathfrak{L}(S, U) \longrightarrow \mathrm{Ext}(\ell_\infty/S, U) \longrightarrow \mathrm{Ext}(\ell_\infty, U).$$

Since $\mathrm{Ext}(\ell_\infty, U) = 0$, one obtains that every exact sequence $0 \to U \to X \to \ell_\infty/S \to 0$ fits in a push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & \ell_\infty/S & \longrightarrow & 0.
\end{array}$$

Since U is universally separably injective, the lower sequence splits.

The converse is clear: every operator $t : S \to U$ from a separable Banach space into a space U produces a push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/S & \longrightarrow & 0 \\
 & & {\scriptstyle t}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U & \longrightarrow & \mathrm{PO} & \longrightarrow & \ell_\infty/S & \longrightarrow & 0.
\end{array}$$

The lower sequence splits by the assumption $\mathrm{Ext}(\ell_\infty/S, U) = 0$ and so t extends to ℓ_∞, according to the splitting criterion for push-out sequences. $\square$

This leads to the unexpected proposition given below.

Proposition 5.6. *Every short exact sequence* $0 \to \ell_\infty/c_0 \to X \to \ell_\infty/c_0 \to 0$ *splits, that is,* $\mathrm{Ext}(\ell_\infty/c_0, \ell_\infty/c_0) = 0$.

Essentially, only three solutions of the equation $\mathrm{Ext}(X, X) = 0$ were previously known, namely: the injective spaces (by the very definition), c_0 (by the Sobczyk theorem), and the spaces $L_1(\mu)$ (by Lindenstrauss' lifting).

It is not however true that $\mathrm{Ext}(U, U) = 0$ for all universally separably injective spaces U: if V is a universally separably injective non-injective space (such as $\ell_\infty^c(\Gamma)$ or the ultraproducts appearing in Theorem 4.10 or Corollary 4.11) then every exact sequence $0 \to V \to \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)/V \to 0$ is not trivial. By Proposition 3.7 $W = \ell_\infty(\Gamma)/V$ is universally separably injective and, obviously, $\mathrm{Ext}(W, V) \neq 0$. The product space $U = V \times W$ is universally separably injective and $\mathrm{Ext}(U, U) \neq 0$.

In rough contrast with Proposition 5.3, one has the following.

Corollary 5.7. *Rosenthal's property (V) is not a* 3*-space property.*

Proof. With the same construction as above, start with a nontrivial exact sequence $0 \to \ell_2 \to E \to \ell_\infty \to 0$ (see [20, Section 4.2]) and construct an exact sequence

$$0 \longrightarrow \ell_\infty(\Gamma, \ell_2) \longrightarrow X \xrightarrow{q} \ell_\infty \longrightarrow 0,$$

where q cannot be an isomorphism on a copy of ℓ_∞. So X lacks Rosenthal's property (V). The space $\ell_\infty(\Gamma, \ell_2)$ has Rosenthal's property (V) as a quotient of $\ell_\infty(\Gamma, \ell_\infty) = \ell_\infty(\Gamma \times \mathbb{N})$, since the property obviously passes to quotients. $\square$

6. 1-separably injective spaces

Our first result here establishes a major difference between 1-separably injective and general separably injective spaces: 1-separably injective spaces must be Grothendieck (hence they cannot be separable or WCG) while a 2-separably injective space, such as c_0, can be even separable. The following lemma due to Lindenstrauss [43, p. 221, proof of (i) $\Rightarrow$ (v)] provides a quite useful technique.

Lemma 6.1. *Let E be a 1-separably injective space and Y a separable subspace of X, with* $\operatorname{dens} X = \aleph_1$. *Then every operator $t : Y \to E$ can be extended to an operator $T : X \to E$ with the same norm.*

This yields the following.

Proposition 6.2. *Under* CH *every 1-separably injective Banach space is universally 1-separably injective and therefore a Grothendieck space.*

Proof. Let E be 1-separably injective, X an arbitrary Banach space and $t : Y \to E$ an operator, where Y is a separable subspace of X. Let $[t(Y)]$ be the closure of the image of t. This is a separable subspace of E and so there is an isometric embedding $u : [t(Y)] \to \ell_\infty$. As ℓ_∞ is 1-injective there is an operator $T : X \to \ell_\infty$ whose restriction to Y agrees with ut. Thus it suffices to extend the inclusion of $[t(Y)]$ into E to ℓ_∞. But, under CH, the density character of ℓ_∞ is $\aleph_1$ and the preceding Lemma applies. The 'therefore' part is now a consequence of Proposition 3.5(b). $\square$

The "therefore" part survives in ZFC.

Proposition 6.3. *Every 1-separably injective space is a Grothendieck and a Lindenstrauss space.*

Proof. The proof of Proposition 3.5 yields that 1-separably injective spaces are of type $\mathcal{L}_{\infty,1^+}$, that is, Lindenstrauss spaces; indeed, during that proof it was shown that the bidual of a 1-separably injective space E is 1-injective, hence a Lindenstrauss space as well as E itself. It remains to prove that a 1-separably injective space E must be Grothendieck. It suffices to show that c_0 is not complemented in E, so let $\jmath : c_0 \longrightarrow E$ be an embedding. Consider an almost-disjoint family $\mathscr{M}$ of size $\aleph_1$ formed by infinite subsets of $\mathbb{N}$ and construct the associated Johnson–Lindenstrauss twisted sum space

$$0 \longrightarrow c_0 \longrightarrow JL_\infty(\mathscr{M}) \longrightarrow c_0(\aleph_1) \longrightarrow 0.$$

The space $JL_\infty(\mathscr{M})$ has density character $\aleph_1$ and, by Lemma 6.1, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & c_0 \arrow[r] \arrow[d, equal] & JL_\infty(\mathscr{M}) \arrow[r] \arrow[d] & c_0(\aleph_1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & c_0 \arrow[r, "\jmath"] & E \arrow[r] & E/\jmath(c_0) \arrow[r] & 0. \end{tikzcd}$$

If c_0 was complemented in E then it would be complemented in $JL_\infty(\mathscr{M})$ as well, which is not. $\square$

Proposition 6.2 leads to the question about the necessity of the continuum hypothesis. We will prove now that it cannot be dropped.

6.1. A 1-separably injective but not 1-universally separably injective $C(K)$

Lemma 6.4. *Let K, L, M be compact spaces and let $f : K \longrightarrow M$ be a continuous map, with $\jmath = f^\circ : C(M) \longrightarrow C(K)$ its induced operator, and let $\imath : C(M) \longrightarrow C(L)$ be a positive norm one operator. Suppose that $S : C(L) \longrightarrow C(K)$ is an operator with $\|S\| = 1$ and $S\imath = \jmath$. Then S is a positive operator.*

Proof. Obviously $S \geq 0$ if and only if $S^*\delta_x \geq 0$ for all $x \in K$, where δ_x is the unit mass at x and $S^* : C(K)^* \to C(L)^*$ is the adjoint operator. Fix $x \in K$. By the Riesz theorem we have that $S^*\delta_x = \mu$ is a measure of total variation $\|\mu\| \leq 1$. Let $\mu = \mu^+ - \mu^-$ be the Hahn–Jordan decomposition of μ, so that $\|\mu\| = \|\mu^+\| + \|\mu^-\|$, with $\mu^+, \mu^- \geq 0$. We have that $\delta_{f(x)} = \jmath^*\delta_x = \imath^* S^*\delta_x = \imath^* \mu$; thus

$$\delta_{f(x)} = \imath^* \mu^+ - \imath^* \mu^- \quad \text{and} \quad \|\delta_{f(x)}\| = \|\imath^* \mu^+\| + \|\imath^* \mu^-\|.$$

Since $\imath$ is a positive operator these imply that the above is the Hahn–Jordan decomposition of $\delta_{f(x)}$ and so $\imath^* \mu^- = 0$, hence $\mu^- = 0$. $\square$

Definition 6.5. Let L be a zero-dimensional compact space. An $\aleph_2$-Lusin family on L is a family $\mathscr{F}$ of pairwise disjoint nonempty clopen subsets of L with $|\mathscr{F}| = \aleph_2$, such that whenever $\mathscr{G}$ and $\mathscr{H}$ are subfamilies of $\mathscr{F}$ with $|\mathscr{G}| = |\mathscr{H}| = \aleph_2$, then

$$\overline{\bigcup\{G \in \mathscr{G}\}} \cap \overline{\bigcup\{G \in \mathscr{H}\}} \neq \varnothing.$$

The following lemma shows the consistency of the existence of an $\aleph_2$-Lusin family on $\mathbb{N}^*$. This is rather folklore of set-theory, but we did not find a reference so we state it and give a hint of the proof.

Lemma 6.6. *Under MA and the assumption $\mathfrak{c} = \aleph_2$ there exists an $\aleph_2$-Lusin family on $\mathbb{N}^*$.*

Proof. By Stone duality, since the Boolean algebra associated to $\mathbb{N}^*$ is $\wp(\mathbb{N})/\text{fin}$, an $\aleph_2$-Lusin family on $\mathbb{N}^*$ is all the same as an almost disjoint family $\{A_\alpha\}_{\alpha<\omega_2}$ of infinite subsets of $\mathbb{N}$ such that for every $B \subset \mathbb{N}$ either $\{\alpha : |A_\alpha \setminus B| \text{ is finite}\}$ or $\{\alpha : |A_\alpha \cap B| \text{ is finite}\}$ has cardinality $< \aleph_2$. Let $\{B_\alpha : \alpha < \omega_2\}$ be an enumeration of all infinite subsets of $\mathbb{N}$. We construct the sets A_α inductively on α. Suppose A_γ has been constructed for $\gamma < \alpha$. We define a forcing notion $\mathbb{P}$ whose conditions are pairs $p = (f_p, F_p)$ where f_p is a $\{0, 1\}$-valued function on a finite subset $\text{dom}(f_p)$ of $\mathbb{N}$ and F_p is a finite subset of α. The order relation is that $p < q$ if f_p extends f_q, $F_p \supset F_q$ and f_p vanishes in $A_\gamma \setminus \text{dom}(f_q)$ for $\gamma \in F_q$. One checks that this forcing is ccc. Hence, by MA, using a big enough generic filter the forcing provides an infinite set $A_\alpha \subset \mathbb{N}$ such that, for all $\gamma < \alpha$,

(1) $A_\alpha \cap A_\gamma$ is finite, and
(2) if B_γ is not contained in any finite union of A_δ's, then $A_\alpha \cap B_\gamma$ is infinite. $\square$

Theorem 6.7. *It is consistent that there exists a compact space K for which the Banach space $C(K)$ is* 1*-separably injective but not universally* 1*-separably injective.*

Proof. We will suppose that $\mathfrak{c} = \aleph_2$ and that there exists an $\aleph_2$-Lusin family in $\mathbb{N}^*$. Under these hypotheses, let K be the Stone dual compact space of the Cohen–Parovičenko Boolean algebra of [30, Theorem 5.6]. The definition of that Boolean algebra implies that K is an F-space and thus

$C(K)$ is 1-separably injective by Proposition 4.2. We show that it is not universally 1-separably injective. The argument follows the scheme of [30, Theorem 5.10], where they prove that K does not map onto $\beta\mathbb{N}$, but we use $\aleph_2$-Lusin families instead of ω_2-chains because they fit better in the functional analytic context.

Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of pairwise disjoint clopen subsets of K, and let $U = \bigcup_n U_n$. Let $c \subset \ell_\infty$ be the Banach space of convergent sequences, and $t : c \longrightarrow C(K)$ be the operator given by $t(z)(x) = z_n$ if $x \in U_n$ and $t(z)(x) = \lim z_n$ if $x \notin U$.

If $C(K)$ were universally 1-separably injective, we should have an extension $T : \ell_\infty \longrightarrow C(K)$ of t with $\|T\| = 1$. We shall derive a contradiction from the existence of such operator.

Notice that the conditions of Lemma 6.4 are applied, so T is positive (observe that $c = C(\mathbb{N} \cup \{\infty\})$ and $T = f^\circ$ where $f : K \longrightarrow \mathbb{N} \cup \{\infty\}$ is given by $f(x) = n$ if $x \in U_n$ and $f(x) = \infty$ if $x \notin U$).

For every $A \subset \mathbb{N}$ we will denote $[A] = \overline{A}^{\beta\mathbb{N}} \setminus \mathbb{N}$. The clopen subsets of $\mathbb{N}^*$ are exactly the sets of the form $[A]$, and we have that $[A] = [B]$ if and only if $(A \setminus B) \cup (B \setminus A)$ is finite.

Let $\mathscr{F}$ be an $\aleph_2$-Lusin family in $\mathbb{N}^*$. For $F = [A] \in \mathscr{F}$ and $0 < \varepsilon < \frac{1}{2}$, let

$$F_\varepsilon = \{x \in K \setminus U : T(1_A)(x) > 1 - \varepsilon\}.$$

Let us remark that F_ε depends only on F and not on the choice of A. This is because if $[A] = [B]$, then $1_A - 1_B \in c_0$; hence $T(1_A - 1_B) = t(1_A - 1_B)$ which vanishes out of U, so $T(1_A)|_{K\setminus U} = T(1_B)|_{K\setminus U}$.

Claim 1. *If $\delta < \varepsilon$ and $F \in \mathscr{F}$, then $\overline{F_\delta} \subset F_\varepsilon$.*

Claim 2. *$F_\varepsilon \cap G_\varepsilon = \varnothing$ for every $F \neq G$.*

Proof of Claim 2. Since $F \cap G = \varnothing$ we can choose $A, B \subset \mathbb{N}$ such that $F = [A]$, $G = [B]$ and $A \cap B = \varnothing$. If $x \in F_\varepsilon \cap G_\varepsilon$, $\bar{T}(1_A + 1_B)(x) > 2 - 2\varepsilon > 1$ which is a contradiction because $1_A + 1_B = 1_{A\cup B}$ and $\|\bar{T}(1_{A\cup B})\| \leq \|\bar{T}\|\|1_{A\cup B}\| = 1$. □

For every $F \in \mathscr{F}$, let $\tilde{F}$ be a clopen subset of $K \setminus U$ such that $\overline{F_{0.2}} \subset \tilde{F} \subset F_{0.3}$. By the preceding claims, this is a disjoint family of clopen sets. It follows from Proposition 2.6 and Corollary 5.12 in [30] that $K \setminus U$ does not contain any $\aleph_2$-Lusin family. Therefore we can find $\mathscr{G}, \mathscr{H} \subset \mathscr{F}$ with $|\mathscr{G}| = |\mathscr{H}| = \aleph_2$ such that

$$\overline{\bigcup\{\tilde{G} : G \in \mathscr{G}\}} \cap \overline{\bigcup\{\tilde{H} : H \in \mathscr{H}\}} = \varnothing.$$

Now, for every $n \in \mathbb{N}$ choose a point $p_n \in U_n$. Let $g : \beta\mathbb{N} \longrightarrow K$ be a continuous function such that $g(n) = p_n$.

Claim 3. *For $u \in \beta\mathbb{N}$, $A \subset \mathbb{N}$, $T(1_A)(g(u)) = \begin{cases} 1, & \text{if } u \in [A]; \\ 0, & \text{if } u \notin [A]. \end{cases}$*

Proof of Claim 3. It is enough to check it for $u = n \in \mathbb{N}$. This is a consequence of the fact that T is positive, because if $m \in A$, $n \notin A$, then $0 \leq t(1_m) \leq T(1_A) \leq t(1_{\mathbb{N}\setminus\{n\}}) \leq 1$. □

The function g is one-to-one because

$$\overline{\{p_n : n \in A\}} \cap \overline{\{p_n : n \notin A\}} = \varnothing$$

for every $A \subset \mathbb{N}$, as the function $T(1_A)$ separates these sets. On the other hand, as a consequence of Claim 3 above, for every $F \in \mathscr{F}$ and every ε, $g^{-1}(F_\varepsilon) = F$, and also $g^{-1}(\tilde{F}) = F$. These facts make the families $\mathscr{H}$ and $\mathscr{G}$ above to contradict that $\mathscr{F}$ is an $\aleph_2$-Lusin family in $\mathbb{N}^*$. □

7. Concluding remarks

7.1. An application to duality

Now, we give an application which uses both ultraproducts and M-ideals. As a preparation, let us check the following.

Lemma 7.1. *Let $(E_i)_{i\in I}$ be a family of Banach spaces and $\mathscr{U}$ an ultrafilter on I. Then $c_0^{\mathscr{U}}(E_i)$ is an M-ideal in $\ell_\infty(E_i)$.*

Proof. It is very difficult to manage the dual of $\ell_\infty(E_i)$ and so we need a different approach avoiding duality. It is proved in [33, Theorem 2.2] that J is an M-ideal in X if and only if it satisfies the following condition: given a finite family of closed balls $B(x^k, r_k)$ in X such that $B(x^k, r_k) \cap J \neq \varnothing$ for all k and

$$\bigcap_k B(x^k, r_k) \neq \varnothing,$$

one has

$$\bigcap_k B(x^k, r_k + \varepsilon) \cap J \neq \varnothing$$

for each $\varepsilon > 0$.

Let us check this condition for $c_0^{\mathscr{U}}(E_i)$. Let $B(x^k, r_k)$ be the corresponding balls and take $x = (x_i)$ in their intersection. Also, for each k, pick $y^k \in B(x^k, r_k) \cap c_0^{\mathscr{U}}(E_i)$.

Now, given $\varepsilon > 0$, as $\|y_i^k\| \to 0$ along $\mathscr{U}$ we may find I_ε in $\mathscr{U}$ such that $\|y_i^k\| \le \varepsilon$ for all k and all $i \in I_\varepsilon$. We define $y = (y_i)$ taking

$$y_i = \begin{cases} 0 & \text{for } i \in I_\varepsilon \\ x_i & \text{otherwise.} \end{cases}$$

It is clear that $y \in \bigcap_k B(x^k, r_k + \varepsilon) \cap c_0^{\mathscr{U}}(E_i)$. $\square$

A Banach space E has the Uniform Approximation Property (UAP) if and only if all its ultrapowers have the BAP. This is not the genuine definition, but an equivalent condition (see [23, Section 7]). One has the following.

Proposition 7.2. *Let X be a separable Banach space and let E be a Banach space. Suppose either X has the BAP or E has the UAP. If $\operatorname{Ext}(X, E) = 0$ then $\operatorname{Ext}(X, E_{\mathscr{U}}) = 0$ for all ultrafilters $\mathscr{U}$.*

Proof. We write the proof in the case where X has the BAP and leave the case where E has the UAP to the reader. Let $\pi : \ell_1 \to X$ be any quotient map and set $K = \ker \pi$. It is well-known that, given a Banach space Y, the condition $\operatorname{Ext}(X, Y) = 0$ is equivalent to: every operator $v : K \to Y$ has an extension $\tilde{v} : \ell_1 \to Y$. If so, this can be done with $\|\tilde{v}\| \le C\|v\|$, by the open mapping theorem.

So, let $u : K \to E_{\mathscr{U}}$ be an operator. We know from Lusky [45] that K has the BAP when X has the BAP. In view of Lemma 7.1, Theorem 4.5 (1) applies and so u lifts to an operator $\tilde{u} : K \to \ell_\infty(I, E)$ we may write $\tilde{u} = (u_i)$, with $u_i \in L(K, E)$. As $\operatorname{Ext}(X, E)$ vanishes we can extend each u_i to an operator $v_i : \ell_1 \to E$, with $\|v_i\| \le C\|u_i\| \le C\|\tilde{u}\|$. This gives an operator $\ell_1 \to \ell_\infty(I, E)$ and composing with the quotient map defining $E_{\mathscr{U}}$ we get the required extension $v : \ell_1 \to E_{\mathscr{U}}$. $\square$

It would be interesting to know if the BAP (or the UAP) is truly necessary in Proposition 7.2. The separability of X is necessary since ultraproducts via countably incomplete ultrafilters are never injective (see [35, Theorem 2.6] or [55, Section 8]; but also [8]). Thus, there must be some Banach space X for which $\mathrm{Ext}(X, (\ell_\infty)_{\mathscr{U}}) \neq 0$, while one obviously has $\mathrm{Ext}(X, \ell_\infty) = 0$. An unexpected consequence of Proposition 7.2 is the following.

Corollary 7.3. *Let X be a separable Banach space and Y a Banach space. Suppose either X has the BAP or Y has the UAP. Then* $\mathrm{Ext}(X, Y) = 0$ *implies* $\mathrm{Ext}(Y^*, X^*) = 0$.

Proof. Since $\mathrm{Ext}(X, Y^*) = \mathrm{Ext}(Y, X^*)$ then $\mathrm{Ext}(Y^*, X^*) = \mathrm{Ext}(X, Y^{**})$ for all Banach spaces X and Y. Now, if $\mathrm{Ext}(X, Y) = 0$ we know from Proposition 7.2 that $\mathrm{Ext}(X, Y_{\mathscr{U}}) = 0$ for all ultrafilters $\mathscr{U}$. Since Y^{**} is a complemented subspace of a suitable ultrapower of Y, the result follows. $\square$

7.2. *Open problems*

There are many questions that arise naturally from the content of this paper and we have been unable to resolve.

It would be interesting to characterize (universally) separably injective $C(K)$ in terms of topological properties of K. As for concrete examples we do not know if separable injectivity passes to injective tensor products and, in fact, we do not know whether $\ell_\infty \check{\otimes}_\varepsilon \ell_\infty = C(\beta\mathbb{N} \times \beta\mathbb{N})$ is separably injective or not.

We do not know if 1-separably injective spaces X must be universally separably injective in ZFC even if $X = C(K)$, with K an F-space. See Proposition 6.2 and Theorem 6.7.

It is an open question to decide if $\mathrm{Ext}(X, Y) = 0$ implies $\mathrm{Ext}(Y^*, X^*) = 0$ for all Banach spaces X and Y. The main difficulty for a direct proof is that, as it was shown in [19, Theorem 5], there are elements in $\mathrm{Ext}(Y^*, X^*)$ which are not duals of elements of $\mathrm{Ext}(X, Y)$.

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